4) (a) Atomic hydrogen contains 5.5×10^{25} atoms/m^3 at a certain temperature and pressure. When an electric field of 4 kV/m is applied, each dipole formed by the electron and positive nucleus has an effective length of 7.1×10^{-19} m. Find

1. The net dipole moment (P). **Solution (2 Marks)**
2. The dielectric constant (ε_r). **Solution (2 Marks)**

Solution (4 Marks)

1) Find P: With all identical dipoles, we have

$P = Nqd = (5.5\times10^{25})(1.602\times10^{-19})(7.1\times10^{-19}) = 6.26\times10^{-12}$ C/m^2 = 6.26 pC/m^2

2) Find ε_r: We use $P = \varepsilon_0 \chi_e E$, and so

$$\chi_e = \frac{P}{\varepsilon_0 E} = \frac{6.26 \times 10^{-12}}{(8.85 \times 10^{-12})(4 \times 10^3)} = 1.76 \times 10^{-4}$$

$$\therefore \varepsilon_r = 1 + \chi_e = 1.000176.$$

(b) For a point charge Q = 25 nC lies at (3,4,6)

1. Find $\overline{E}$ at (2,1,0). **Solution (3 Marks)**
2. Find ρ_s at (2,1,0) when a grounded conducting plate is places at z = 0. **Solution (3 Marks)**

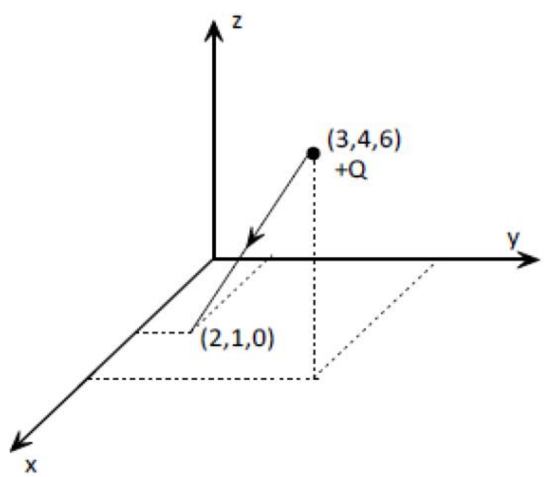
1) Find $\overline{E}$

$$\overline{E} = \frac{Q}{4\pi\varepsilon R_1^2}\overline{a}_{R1}$$

$$\overline{R}_1 = -\overline{a}_x - 3\overline{a}_y - 6\overline{a}_z$$

$$R_1 = \sqrt{1+9+36} = \sqrt{46}$$

$$\overline{a}_{R1} = \frac{-\overline{a}_x - 3\overline{a}_y - 6\overline{a}_z}{\sqrt{46}}$$



$$\overline{E} = \frac{25 \times 10^{-9}}{4\pi(8.85 \times 10^{-12})(46)}\left[\frac{-\overline{a}_x - 3\overline{a}_y - 6\overline{a}_z}{\sqrt{46}}\right]$$

$$\overline{E} = -0.72\overline{a}_x - 2.16\overline{a}_y - 4.32\overline{a}_z$$

2) Using the method of images, we will replace the point charge + Q at (3,4,6) with a grounded conducting plate by two point charges +Q at (3,4,6) and -Q at (3,4,-6).

$$\overline{E} = \frac{Q}{4\pi\varepsilon R_1^2}\overline{a}_{R1} + \frac{-Q}{4\pi\varepsilon R_2^2}\overline{a}_{R2}$$

$$\overline{R}_2 = -\overline{a}_x - 3\overline{a}_y + 6\overline{a}_z$$

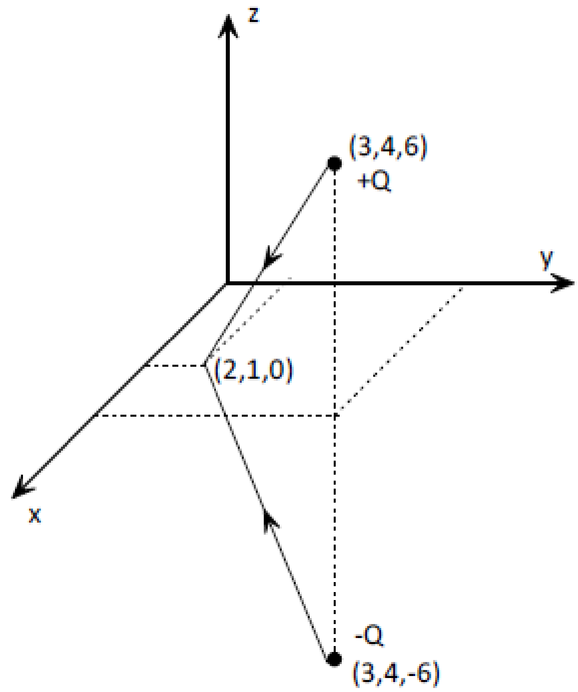
$$R_2 = \sqrt{1+9+36} = \sqrt{46}$$

$$\overline{a}_{R2} = \frac{-\overline{a}_x - 3\overline{a}_y + 6\overline{a}_z}{\sqrt{46}}$$

$$\overline{E} = \frac{Q}{4\pi\varepsilon R_1^2}[\overline{a}_{R1} - \overline{a}_{R2}]$$

$$\overline{E} = \frac{25\times10^{-9}}{4\pi(8.85\times10^{-12})(46)}\left[\frac{-12}{\sqrt{46}}\overline{a}_z\right]$$

$$\overline{E} = -8.64\overline{a}_z$$



$$\rho_S = |D_n| = |\varepsilon E_n| = \varepsilon(8.64) = 0.076\ \text{nC/m}^2$$

(c) Two perfect dielectrics have relative permittivities $\varepsilon_{r1} = 2$ and $\varepsilon_{r2} = 8$. The planar interface between them is the surface x – y + 2z = 5. The origin lies in region 1. If $E_1 = 100\hat{a}_x + 200\hat{a}_y - 50\hat{a}_z$ V/m, find E_2. **Solution (5 Marks)**

We need to find the components of E1 that are normal and tangent to the boundary, and then apply the appropriate boundary conditions. The normal component will be $E_{N1} = E_1 \cdot n$. Taking f = x − y + 2z, the unit vector that is normal to the surface is

$$\text{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{6}}[\overline{a}_x - \overline{a}_y + 2\overline{a}_z]$$

$$E_{\text{N1}} = E_1.\text{n} = \frac{1}{\sqrt{6}}[\overline{a}_x - \overline{a}_y + 2\overline{a}_z].[100\overline{a}_x + 200\overline{a}_y - 50\overline{a}_z] = -81.7\ \text{V/m}$$

Since the magnitude is negative, the normal component points into region 1 from the surface. Then

$$E_{\text{N1}} = E_1.\text{n} = -81.7\left(\frac{1}{\sqrt{6}}\right)[\overline{a}_x - \overline{a}_y + 2\overline{a}_z]$$

$$E_{\text{N1}} = -33.3\overline{a}_x + 33.3\overline{a}_y - 66.67\overline{a}_z\ \text{V/m}$$

Now, the tangential component will be

$$E_{T1} = E_1 - E_{N1} = 133.3\overline{a}_x + 166.7\overline{a}_y + 16.67\overline{a}_z \text{ V/m}$$

Our boundary conditions state that $E_{T2} = E_{T1}$ and $E_{N2} = (\varepsilon_{r1}/\varepsilon_{r2})E_{N1} = (1/4)E_{N1}$.

Thus, $E_2 = E_{T2} + E_{N2} = E_{T1} + \frac{1}{4}E_{N1} = 125\overline{a}_x + 175\overline{a}_y$ V/m.

5) (a) The potential $V = 2x + 4y - 2z$ volt exists in free surrounding a perfectly conducting surface. Point P(4,3,2) lies on the surface.

1. Give the equation of the surface. **Solution (3 Marks)**

Since P lies on the conductor surface, the potential at P is

$$V = 2x + 4y - 2z$$

$$V = 2(4) + 4(3) - 2(2) = 16 \text{ V}$$

Since the conductor is an equipotential surface, all points on the surface have potential of 16 V, so the equation of the surface will be

$$2x + 4y - 2z = 16$$

2. Find the unit vector normal to the surface at P. **Solution (3 Marks)**

To find the unit vector normal to the surface at P, it is a unit vector in the direction of the electric field because the electric field is normal to the surface

$$\overline{E} = -\nabla V$$

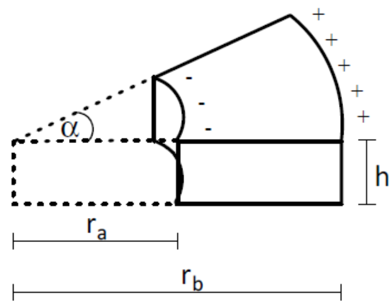
$$\overline{E} = -\left(\frac{\partial V}{\partial x}\overline{a}_x + \frac{\partial V}{\partial y}\overline{a}_y + \frac{\partial V}{\partial z}\overline{a}_z\right)$$

$$\overline{E} = -2\overline{a}_x - 4\overline{a}_y + 2\overline{a}_z$$

$$\overline{a}_E = \frac{-2\overline{a}_x - 4\overline{a}_y + 2\overline{a}_z}{\sqrt{24}}$$

$$\overline{a}_E = \frac{-\overline{a}_x - 2\overline{a}_y + \overline{a}_z}{\sqrt{6}}$$

(b) Find the capacitance between the curved plates shown in the figure. **Solution (6 Marks)**



Applying Gauss's law

$$\oiint \overline{D}.\overline{ds} = Q_{encl}$$

$$D(r\alpha h) = \rho_S(r_a \alpha h)$$

$$D = \frac{\rho_S r_a}{r}(-\overline{a}_r)$$

$$E = \frac{\rho_S r_a}{\varepsilon r}(-\overline{a}_r)$$

$$V = -\int_{-}^{+} \overline{E}.\overline{dl} = -\int_{r_a}^{r_b} \frac{\rho_S r_a}{\varepsilon r}(-\overline{a}_r).dr\overline{a}_r = \int_{r_a}^{r_b} \frac{\rho_S r_a}{\varepsilon r} dr$$

$$V = \frac{\rho_S r_a}{\varepsilon}\int_{r_a}^{r_b} \frac{dr}{r} = \frac{\rho_S r_a}{\varepsilon} \ln r|_{r_a}^{r_b}$$

$$\therefore V = \frac{\rho_S r_a}{\varepsilon}\left[\ln \frac{r_b}{r_a}\right]$$

$$C = \frac{Q}{V} = \frac{\rho_S r_a \alpha h}{\frac{\rho_S r_a}{\varepsilon}\left[\ln \frac{r_b}{r_a}\right]}$$

$$\therefore C = \frac{\varepsilon \alpha h}{\ln \frac{r_b}{r_a}} \text{ F}$$

6) a) Discuss briefly Gauss' Law for the magnetic field, and then compare it with that of the electric field. **Solution (4 Marks)**

If the flux is evaluated through a closed surface, we have in the case of electric flux, Gauss' Law:

$$\Psi_{net} = \oint_s \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

the same were to be done with magnetic flux density, we would find:

$$\Phi_{net} = \oint_s \mathbf{B} \cdot d\mathbf{S} = 0$$

The implication is that (for our purposes) there are no magnetic charges -- specifically, no point sources of magnetic field exist. A hint of this has already been observed, in that magnetic field lines always close on themselves.

b) A current filament carrying 15 A in the $\boldsymbol{a}_z$ direction lies along the entire *z* axis. Find **H** in rectangular coordinates at point P (2,-4, 4). **Solution (6 Marks)**

Along z-axis

From the figure, we have $\overline{R} = 2\overline{a}_x - 4\overline{a}_y$

$$\overline{H} = \frac{I}{4\pi R}(\sin\alpha_2 - \sin\alpha_1)\overline{a}_H$$

$$= \frac{15}{4\pi(\sqrt{4+16})}(\sin(90) - \sin(-90))\overline{a}_H$$

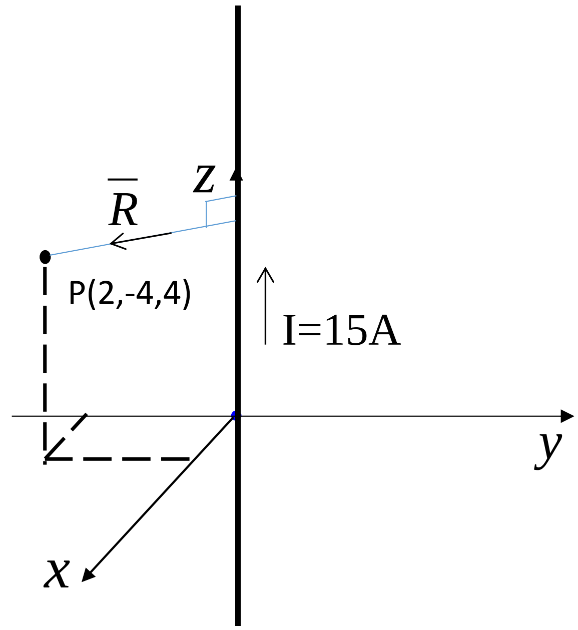
$$= \frac{15}{2\pi(\sqrt{20})}\overline{a}_H \quad \text{A/m}$$

$$\hat{a}_H = \hat{a}_z \times \hat{a}_R$$

$$= \hat{a}_z \times \frac{\overline{R}}{|\overline{R}|} = \hat{a}_z \times \frac{2\overline{a}_x - 4\overline{a}_y}{\sqrt{20}} = \frac{2\overline{a}_y + 4\overline{a}_x}{\sqrt{20}}$$

$$\overline{H} = \frac{15}{2\pi(\sqrt{20})} * \frac{2\overline{a}_y + 4\overline{a}_x}{\sqrt{20}} \quad \text{A/m}$$

$$\therefore \overline{H} = 0.477\overline{a}_x + 0.239\overline{a}_y \quad \text{A/m}$$



c) Define the self-inductance, then derive an expression for the self-inductance of a long solenoid of *N* turns, radius *a*, and length *L*.

Solution (2 Marks)

- **We can define the inductance (or self-inductance) as the ratio of the total flux linkages to the current which they link,**

$$L = \frac{\Lambda}{I} = \frac{N\Phi}{I} \quad \text{Henry}$$

where Λ (lambda) is the total flux linkage of the inductor

Solution (6 Marks)

Assume all the flux Φ links all N turns and that $\overline{B}$ does not vary over the cross section area of the solenoid

$$\Lambda = \Phi N = B\left(\pi a^2\right)N$$

We have $\overline{B} = \mu \overline{H}$

$$\Lambda = \left(\mu H\right)\left(\pi a^2\right)N = \left(\frac{\mu N I}{l}\right)\left(\pi a^2\right)N$$

$$= \left(\frac{\mu N^2 I}{l}\right)\left(\pi a^2\right)$$

$$\therefore L = \frac{\Lambda}{I} = \frac{\mu N^2 \pi a^2}{l}$$